HEATING OF A BODY IN A BOUNDED VOLUME OF WELL MIXED FLUID

M. D. Mikhailov

UDC 536.242

The problem of the nonstationary heating of an arbitrary body of finite dimensions in a bounded fluid volume is solved for an arbitrary volume heat source and an arbitrary initial-temperature distribution.

In the classical theory of heat conduction [1] the temperature of the ambient medium $T_f(\tau)$ is usually given as a function of time. In a number of cases, however, it is necessary to relate its variation with changes in the temperature of the heated body by means of the heat-balance equation.

This approach was adopted in [2, 3], where it was shown that the problem of determining the temperature field of a solid particle in a parallel or counter-flow [4-9] and the problem of determining the kinetics of periodic parallel and counter-flow extraction from porous particles [10] are identical with the problem of a body heated in a bounded volume of well mixed fluid.

Below, this problem is solved for an arbitrary body of finite dimensions. The temperature field of the body is described by the equation

$$\frac{\partial T(M,\tau)}{\partial \tau} = a\Delta T(M,\tau) + \frac{w(M,\tau)}{c\gamma}, \tag{1}$$

where $\tau > 0$; $M \in V$; $N \in S$; V is a certain region bounded by the closed surface S; n is the outward normal to S.

At the initial instant $\tau = 0$, a body with temperature

$$T(M,0) = f_0(M) \tag{2}$$

is placed in a bounded fluid volume V_f at the initial temperature

$$T_f(0) = T_c. \tag{3}$$

In addition to the usual boundary condition of the third kind

$$-\lambda \frac{\partial T(N,\tau)}{\partial n} + \alpha(N)\{T_f(\tau) - T(N,\tau)\} = 0, \tag{4}$$

from the heat-balance equation for the fluid we obtain

$$\lambda \oint \frac{\partial T(N,\tau)}{\partial n} dS + c_f\gamma_f V_f \frac{dT_f(\tau)}{d\tau} = 0, \tag{5}$$

where the parameters of the fluid are denoted by the subscript f.

The solution of the problem in eigenvalues

$$\Delta\psi(M) + \mu^2\psi(M) = 0, \tag{6}$$

$$\psi(N) + \frac{\lambda}{\alpha(N)}\frac{\partial\psi(N)}{\partial n} + \frac{c\gamma}{c_f\gamma_f V_f}\int_V \psi(M)\,dV = 0 \tag{7}$$

is assumed to be known.

Lenin Higher Mechanical and Electrical Engineering Institute, Sofia, Bulgaria. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 19, No. 3, pp. 401-405, September, 1970. Original article submitted March 2, 1970.

Let $\psi_i(M)$ and $\psi_j(M)$ be two eigenfunctions corresponding to different eigenvalues μ_i and μ_j

$$\Delta\psi_i(M) = -\mu_i^2\psi_i(M) \text{ and } \Delta\psi_j(M) = -\mu_j^2\psi_j(M), \tag{8}$$

moreover,

$$\psi_i(N) + \frac{\lambda}{\alpha(N)}\frac{\partial\psi_i(N)}{\partial n} = -\frac{c\gamma}{c_f\gamma_f V_f}\int_V \psi_i(M)\,dV, \tag{9}$$

$$\psi_j(N) + \frac{\lambda}{\alpha(N)}\frac{\partial\psi_j(N)}{\partial n} = -\frac{c\gamma}{c_f\gamma_f V_f}\int_V \psi_j(M)\,dV. \tag{10}$$

Multiplying the first of Eqs. (8) by $\psi_j(M)$, subtracting the second equation mulitplied by $\psi_i(M)$, and integrating the results obtained using the second Green's theorem [11], we find

$$\oint_S \begin{vmatrix} \psi_i(N) & \dfrac{\partial\psi_i(N)}{\partial n} \\ \psi_j(N) & \dfrac{\partial\psi_j(N)}{\partial n} \end{vmatrix} dS = (\mu_i^2 - \mu_j^2)\int_V \psi_i(M)\psi_j(M)\,dV. \tag{11}$$

Treating boundary conditions (9) and (10) as a system of equations in the "unknowns" 1 and $\lambda/\alpha(N)$ and using Cramer's rule to determine the first "unknown" with subsequent integration over the surface of the body, we find

$$\oint_S \begin{vmatrix} \psi_i(N) & \dfrac{\partial\psi_i(N)}{\partial n} \\ \psi_j(N) & \dfrac{\partial\psi_j(N)}{\partial n} \end{vmatrix} dS + (\mu_i^2 - \mu_j^2)\frac{c\gamma}{c_f\gamma_f V_f}\int_V \psi_i(M)\,dV \int_V \psi_j(M)\,dV = 0. \tag{12}$$

From (11) and (12) there follows

$$\int_V \psi_i(M)\left\{\psi_j(M) + \frac{c\gamma}{c_f\gamma_f V_f}\int_V \psi_j(M)\,dV\right\}dV = 0 \quad (i \neq j). \tag{13}$$

An arbitrary, twice-continuously differentiable function can be represented in the series form

$$F(M) = \sum_{i=1}^{\infty} C_i\psi_i(M). \tag{14}$$

In order to determine C_i we multiply both sides of Eq. (14) by $\psi_j(M) + (c\gamma/c_f\gamma_f V_f)\int_V \psi_j(M)dV$ and then integrate over the volume. Then, using (13), we easily find C_i, and series (14) takes the form

$$F(M) = \sum_{i=1}^{\infty} \frac{\displaystyle\int_V F(M)\left\{\psi_i(M) + \frac{c\gamma}{c_f\gamma_f V_f}\int_V \psi_i(M)\,dV\right\}dV}{\displaystyle\int_V \psi_i(M)\left\{\psi_i(M) + \frac{c\gamma}{c_f\gamma_f V_f}\int_V \psi_i(M)\,dV\right\}dV}\,\psi_i(M). \tag{15}$$

In what follows we require the series expansion of the function $F(M) = 1$. For this case from (15) we obtain

$$\sum_{i=1}^{\infty} \frac{\psi_i(M)\displaystyle\int_V \psi_i(M)\,dV}{\displaystyle\int_V \psi_i(M)\left\{\psi_i(M) + \frac{c\gamma}{c_f\gamma_f V_f}\int_V \psi_i(M)\,dV\right\}dV} = \frac{1}{1 + \dfrac{c\gamma V}{c_f\gamma_f V_f}}. \tag{16}$$

In order to solve Eq. (1) with conditions (2)-(5) we use the Laplace integral transform

$$\bar{T}(M, p) = \int_0^{\infty} T(M, \tau)\exp(-p\tau)\,d\tau \tag{17}$$

and the finite integral transform

$$\tilde{T}_i(p) = \int_V \bar{T}(M, p)\left\{\psi_i(M) + \frac{c\gamma}{c_f\gamma_f V_f}\int_V \psi_i(M)\,dV\right\}dV. \tag{18}$$

Comparing (15) and (18), we see that the inversion formula takes the form

$$\bar{T}(M, p) = \sum_{i=1}^{\infty} \frac{\psi_i(M)\tilde{T}_i(p)}{\int_V \psi_i(M)\left\{\psi_i(M) + \frac{c\gamma}{c_f\gamma_f V_f}\int_V \psi_i(M)\,dV\right\}dV}. \tag{19}$$

Applying a Laplace transformation to (1) and multiplying the results obtained by $\psi_i(M) + c\gamma/c_f\gamma_f V_f \int_V \psi_i(M)dV$, subtracting (6) multiplied by $\bar{T}(M, p)$, and integrating over the volume using (18), we obtain

$$\tilde{T}_i(p) = \frac{1}{p + a\mu_i^2}\int_V f_0(M)\psi_i(M)\,dV + \frac{1}{p}\frac{c\gamma}{c_f\gamma_f V_f}\int_V f_0(M)\,dV$$

$$\times \int_V \psi_i(M)\,dV + \frac{a}{p + a\mu_i^2}\oint_S \begin{vmatrix} \psi_i(N) & \frac{\partial\psi_i(N)}{\partial n} \\ \bar{T}(N, p) & \frac{\partial\bar{T}(N,p)}{\partial n} \end{vmatrix} dS + \frac{a}{p}\frac{c\gamma}{c_f\gamma_f V_f}$$

$$\times \int_V \psi_i(M)\,dV \int_V \Delta\bar{T}(M, p)dV + \frac{1}{p + a\mu_i^2}\int \frac{\bar{w}(M, p)}{c\gamma}\psi_i(M)\,dV + \frac{1}{p}\frac{1}{c_f\gamma_f V_f}\int_V \bar{w}(M, p)\,dV \int_V \psi_i(M)\,dV. \tag{20}$$

Applying a Laplace transformation to boundary conditions (4) and (5), we find

$$\bar{T}(N, p) + \frac{\lambda}{\alpha(N)}\frac{\partial\bar{T}(N, p)}{\partial n} = \frac{T_c}{p} - \frac{\lambda}{pc_f\gamma_f V_f}\oint_S \frac{\partial\bar{T}(N, p)}{\partial n}\,dS. \tag{21}$$

As in deriving (12), from (9) and (21) we find

$$\frac{1}{p + a\mu_i^2}\oint_S \begin{vmatrix} \psi_i(N) & \frac{\partial\psi_i(N)}{\partial n} \\ \bar{T}(N, p) & \frac{\partial\bar{T}(N,p)}{\partial n} \end{vmatrix} dS + \frac{c\gamma}{pc_f\gamma_f V_f}\int_V \psi_i(M)\,dV$$

$$\times \int_V \Delta\bar{T}(M, p)\,dV = T_c\left(\frac{1}{p} - \frac{1}{p + a\mu_i^2}\right)\int_V \psi_i(M)\,dV. \tag{22}$$

Substituting (22) in (20) and then the result obtained in inversion formula (19), using (16) we obtain the temperature transform

$$\bar{T}(M, p) = \frac{1}{p}\frac{1}{1 + \frac{c\gamma V}{c_f\gamma_f V_f}}\left\{T_c + \frac{c\gamma}{c_f\gamma_f V_f}\int_V\left[f_0(M) + \frac{\bar{w}(M, p)}{c\gamma}\right]dV\right\}$$

$$+ \sum_{i=1}^{\infty} \frac{\psi_i(M)}{\int_V \psi_i(M)\left\{\psi_i(M) + \frac{c\gamma}{c_f\gamma_f V_f}\int_V \psi_i(M)\,dV\right\}dV}$$

$$\times \frac{1}{p + a\mu_i^2}\left\{\int_V\left[f_0(M) + \frac{\bar{w}(M, p)}{c\gamma}\right]\psi_i(M)\,dV - T_c\int_V \psi_i(M)\,dV\right\}. \tag{23}$$

After inversion we obtain the final solution of the problem

$$T(M, \tau) = \frac{1}{1 + \frac{c\gamma V}{c_f\gamma_f V_f}}\left\{T_c + \frac{c\gamma}{c_f\gamma_f V_f}\int_V f_0(M)\,dV + \frac{1}{c_f\gamma_f V_f}\int_0^{\tau}\int_V w(M, \tau^*)\,dV d\tau^*\right\}$$

$$+\sum_{i=1}^{\infty}\frac{\psi_i(M)\exp(-a\mu_i^2\tau)}{\int\limits_V\psi_i(M)\left\{\psi_i(M)+\frac{c\gamma}{c_f\gamma_fV_f}\int\limits_V\psi_i(M)\,dV\right\}}\left\{\int\limits_V f_0(M)\,\psi_i(M)\,dV\right.$$

$$\left.-T_c\int\limits_V\psi_i(M)\,dV+\int\limits_0^{\tau}\int\limits_V\frac{w(M,\tau^*)}{c\gamma}\psi_i(M)\exp(-\mu_i^2a\tau^*)\,dV d\tau^*\right\}. \qquad (24)$$

For one-dimensional bodies we have

$$\int\limits_V F(M)\,dV=(\Gamma+1)V\int\limits_0^R F(r)\left(\frac{r}{R}\right)^{\Gamma}d\,\frac{r}{R}, \qquad (25)$$

where Γ = 0, 1, 2 for a plate, cylinder, and sphere, respectively.

With the aid of this equation it is easy to obtain from (24) the solutions for a plate, a cylinder, and a sphere presented in [2-10]. Graphs showing the variation of the temperatures of one-dimensional bodies and the temperature of the surrounding medium are given in [2, 3, 7, 8].

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